MEAN VALUE PROPERTIES FOR THE p-LAPLACIAN

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Joint works with:

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INTRODUCTION

Asymptotic expansion for the Laplacian

$$-\Delta u(x) = \underbrace{\frac{C_d}{r^2}\left(u(x) - \fint_{B_r(x)} u(y)\,dy\right)}_{\mathcal{M}_r[u](x)} + O(r^2) \quad \text{as} \quad r \longrightarrow 0^+.$$

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- Zero order approximation

(Consistent) $\mathcal{M}_r[u](x) \xrightarrow{r \to 0^+} -\Delta u(x)$

(Zero order) $\mathcal{M}_r : C(\mathbb{R}^d) \longrightarrow C(\mathbb{R}^d)$

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(Zero order) $\mathcal{M}_r : C(\mathbb{R}^d) \longrightarrow C(\mathbb{R}^d)$

- Preserves the monotonicity of $-\Delta$:

(Monotonicity) At a maximum point x_0 of u

$$\mathcal{M}_r[u](x_0) \geq 0$$

THE p-LAPLACE OPERATOR

$p \in (1, +\infty)$

$$\Delta_p u := \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right)$$
$$= |\nabla u|^{p-2} \left(\Delta u + (p-2) \left\langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right\rangle \right)$$

How to get asymptotic expansions for the p-Laplacian?

Asymptotic expansions for the p-Laplacian (1)

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Conclusion:

Asymptotic expansions for Δ_∞^N (The normalized ∞-Laplacian)

$\Downarrow$

Asymptotic expansions for Δ_p^N (The normalized p-Laplacian)

$\Downarrow$

Asymptotic expansions for p-Harmonic functions.

For a fixed point x, let $v = \frac{\nabla u(x)}{|\nabla u(x)|}$. Then

$$\Delta_\infty^N u(x) = \langle D^2 u(x) v, v \rangle = \frac{u(x+rv) + u(x-rv) - 2u(x)}{r^2} + O(r^2)$$

ASYMPTOTIC EXPANSIONS FOR THE p-LAPLACIAN (2)

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Theorem Manfredi - Parviainen - Rossi (2010)

Let $u \in C^2(\mathbb{R}^d)$, $\alpha = \frac{p-2}{p+d}$, $\beta = \frac{2+d}{p+d}$. Then

$$-\Delta_p^N u(x) = \underbrace{\frac{C_{d,p}}{r^2}\left(u(x) - \frac{\alpha}{2}\left(\sup_{B_r(x)} u + \inf_{B_r(x)} u \right) - \beta \fint_{B_r(x)} u(y)\,dy \right)}_{M_r[u](x)} + o_r(1).$$

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$\mathcal{M}_r$ is a consistent approximation of $-\Delta_p^N$, of zero order, and monotone.

From here, three types of results are desirable

1. CHARACTERIZATION OF VISCOSITY SOLUTIONS

u is a viscosity sol. of $-\Delta_p u = 0$

$$\Updownarrow$$

u is a viscosity sol. of $\mathcal{M}_r[u] = o_r(1)$ as $r \to 0^+$

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② CONVERGENCE OF THE DYNAMIC PROGRAMMING PRINCIPLE

$$(BVP) \begin{cases} -\Delta_p u = 0 & \Omega \\ u = g & \partial\Omega \end{cases} \qquad (S) \begin{cases} \mathcal{M}_r[u_r] = 0 & \Omega \\ u_r = g & \partial\Omega \end{cases}$$

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③ GAME THEORETICAL INTERPRETATION OF (S)

(S) is related to the game "tug-of-war" with noise

The expected value of the game coincides with u_r.

A very natural question arises now:

CAN WE DO SOMETHING SIMILAR FOR NON-HOMOGENEOUS p-LAPLACE PROBLEMS?

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⚠ POSSIBLE STRATEGY:

- To find asymptotic expansions for

$$-\Delta_p$$

(and avoid using the ones for $-\Delta_p^N$)

ASYMPTOTIC EXPANSIONS FOR THE p-LAPLACIAN

del Teso - Lindgren (2021-2022)

$$\boxed{\Delta_p u(x) := \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right)}$$

In dimension $\underline{d=1}$: Let $J_p(\xi) = |\xi|^{p-2}\xi$

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\end{aligned}$$

In dimension $d>1$, similar ideas do not work (to my best knowledge) since Δ_p involves crossed derivatives.

POSSIBLE STRATEGY: use the variational structure of Δ_p

- Δ_p is the first variation of $\boxed{I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx}$

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- Let us approximate $I(u)$ using the vectorial identity $|\vec{a}|^p = \frac{C_{d,p}}{r^p} \fint_{\partial B_r} |\vec{a}\cdot y|^p d\sigma(y)$:

$$I(u) = \frac{C_{d,p}}{p}\int_\Omega \fint_{\partial B_r} \left|\nabla u(x)\cdot \frac{y}{|y|}\right|^p d\sigma(y)\,dx$$

$$\sim \frac{C_{d,p}}{p\, r^p}\int_\Omega \fint_{\partial B_r} |u(x+y)-u(x)|^p d\sigma(y)\,dx =: I_r(u).$$

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- The first variation of I_r is

$$\boxed{\mathcal{H}_r[u](x) = \frac{C_{d,p}}{r^p} \fint_{\partial B_r} J_p(u(x+y)-u(x))\, d\sigma(y).}$$

Theorem (del Teso - Lindgren (2021))

Let $\phi \in C_b^2(B_R(x))$ for some $x \in \mathbb{R}^d$. If $p \in (1,2)$ assume also that $\nabla\phi(x) \neq 0$. Then,

$$\Delta_p \phi(x) = \frac{C_{d,p}}{r^p} \int_{\partial B_r} J_p(\phi(x+y) - \phi(x))\, d\sigma(y) + o_r(1).$$

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We also prove:

- Characterization of viscosity solutions

$$-\Delta_p u = f \iff \mathcal{M}_r[u] = f + o_r(1).$$

- Convergence of the DPP for NON-HOMOGENEOUS p-Laplace problems $\begin{cases} -\Delta_p u = f & \Omega \\ u = g & \partial\Omega \end{cases}$

ASYMPTOTIC EXPANSIONS FOR THE FRACTIONAL p-LAPLACIAN

del Teso - Medina - Ochoa (2024)

THE FRACTIONAL P-LAPLACIAN

$$-(-\Delta)^s_p \phi(x) = \mathrm{P.V.}\int_{|y|>0} J_p(\phi(x+y)-\phi(x))\,\frac{dy}{|y|^{d+sp}}$$

The first (and only?) asymptotic expansion in the literature was

Theorem Bucur - Squassina (2022)

Let $p \geq 2$ and $\phi \in C^2(B_R(x)) \cap L^\infty(\mathbb{R}^d)$. Then

$$-(-\Delta)^s_p \phi(x) = \int_{\mathbb{R}^d \setminus B_r(x)} J(\phi(x+y)-\phi(x))\,\frac{dy}{|y|^{d(p-2)}(|y|^2-r^2)^s} + O(r^{2(1-s)})$$

$$-(-\Delta)_p^s \phi(x) = \text{P.V.}\int_{|y|>0} J_p(\phi(x+y)-\phi(x))\,\frac{dy}{|y|^{d+sp}}$$

Similar computations yield

Theorem

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- The error comes from estimating $\text{P.V.}\int_{B_r} J_p(\phi(x+y)-\phi(x))\frac{dy}{|y|^{d+sp}}$
- At least the error improves with p
- It is not very satisfactory as $s \to 1^-$.

- Instead of just "throwing" the integral on B_r:

$$\text{P.V.}\int_{B_r} J(\phi(x+y)-\phi(x))\,\frac{dy}{|y|^{d+2s}} = \Delta_p\phi(x)\,r^{p(1-s)} + O(r^{\gamma+p(1-s)})$$

- We can now use the asymptotic expansion for the p-Laplacian to produce an asymptotic expansion for the fractional case:

$$\boxed{\begin{aligned}\mathcal{M}_r^{s,p}[\phi](x) := \frac{1}{r^{d+sp}}\int_{B_r} J_p(\phi(x+y)-\phi(x))\,dy \\ + \int_{\mathbb{R}^d\setminus B_r} J_p(\phi(x+y)-\phi(x))\,\frac{dy}{|y|^{d+sp}}\end{aligned}}$$

Theorem del Teso – Medina – Ochoa (2024)

If $p \geq 2$, $s \in (0,1]$ and $\phi \in C^4(B_R(x)) \cap L^\infty(\mathbb{R}^d)$, then

$$-(-\Delta)_p^s \phi(x) = \mathcal{M}_r^{s,p}[\phi](x) + O\left(r^{\gamma + p(1-s)}\right)$$

where

$$\gamma = \begin{cases} 2 & \text{if } p = 2 \text{ or } p \geq 4 \\ p-2 & \text{if } p \in (2,4) \end{cases}$$

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Theorem del Teso – Medina – Ochoa (2024)

If $p > 1$, $s \in (0,1]$ and $\phi \in C^4(B_R(x)) \cap L^\infty(\mathbb{R}^d)$, $\boxed{\nabla \phi \neq 0 \text{ in } B_R(x)}$
then

$$-(-\Delta)_p^s \phi(x) = \mathcal{M}_r^{s,p}[\phi](x) + O(r^{\mu + p(1-s)})$$

where

$$\mu = \begin{cases} 2 & \text{if } p = 2 \text{ or } p \geq 3 \\ p-1 & \text{if } p \in (1,2) \cup (2,3) \end{cases}$$

- These results are optimal (Except in the second theorem in the range $p \in (1,2) \cup (2,3)$).

IS THERE A GAME THEORETICAL INTERPRETATION OF THE p-LAPLACIAN BASED ON THIS ASYMPTOTIC EXPANSION?

$\vdots$

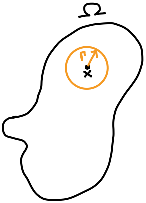
Let us recall what is known

Solutions of $\begin{cases} -\Delta_p u(x) = 0 & \Omega \\ u(x) = g(x) & \partial\Omega \end{cases}$ can be approximated by

$$\text{(DPP)} \begin{cases} u(x) = \dfrac{\beta}{2}\left(\sup\limits_{B_r(x)} u + \inf\limits_{B_r(x)} u \right) + (1-\beta) \displaystyle\fint_{B_r(x)} u(y)\,dy & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases}$$

Problem (DPP) has a probabilistic interpretation:

- Two players play the following game



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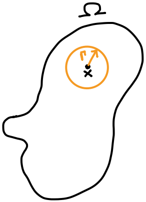

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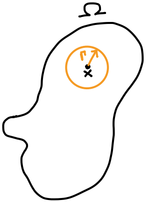

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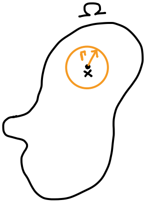

(i) Each turn toss a biased coin with probabilities β (heads), $1-\beta$ (tails)

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(ii)(b) If heads $\longrightarrow$ Tug-of-War in $B_r(x)$

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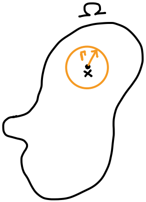

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(i)(a) If tails $\longrightarrow$ Random move in $B_r(x)$

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(ii) Repeat the process until reaching $x_J \in \partial\Omega$
$\longrightarrow$ Player 2 pays Player 1 the value of $g(x_J)$

IS THERE A GAME THEORETICAL INTERPRETATION OF THE p-LAPLACIAN BASED ON THIS ASYMPTOTIC EXPANSION?

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$$\boxed{\alpha+\beta=1}$$

$$\mathcal{M}_r[u](x) = \frac{c_{d,p}}{r^2}\left(u(x) - \text{Average}[u](x)\right)$$

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IS THERE A GAME THEORETICAL INTERPRETATION OF THE p-LAPLACIAN BASED ON THIS ASYMPTOTIC EXPANSION? NO!

$$-\Delta_p^N u(x) = \underbrace{\frac{c_{d,p}}{r^2}\left(u(x) - \frac{\alpha}{2}\left(\sup_{B_r(x)} u + \inf_{B_r(x)} u\right) - \beta \fint_{B_r(x)} u(y)\,dy\right)}_{\mathcal{M}_r[u](x)} + o_r(1).$$

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$$\mathcal{M}_r[u](x) = \frac{c_{d,p}}{r^2}\Big(u(x) - \text{Average}[u](x)\Big)$$

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$$\overline{\mathcal{M}}_r[u](x) \neq \frac{D_{d,p}}{r^p}\Big(u(x) - \text{Average}[u](x)\Big)$$

A GAME-THEORETICAL INTERPRETATION OF THE p-LAPLACIAN

del Teso - Rossi (2025)

Let us consider, for simplicity $Lu = |\nabla u| \Delta u$.

- We want an approximation

$$Lu(x) \sim \frac{1}{\varepsilon^2}\left(\mathcal{A}_\varepsilon[u](x) - u(x)\right)$$

$\hookrightarrow$ Some "average" of u

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⚠ L is 2-homogeneous : $L[\lambda u] = \lambda^2 L[u]$

⚠ $\mathcal{A}_\varepsilon - Id$ is 1-homogeneous : $(\mathcal{A}_\varepsilon - Id)[\lambda u] = \lambda(\mathcal{A}_\varepsilon - Id)[u]$

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⚠ L is 2-homogeneous: $L[\lambda u] = \lambda^2 L[u]$

⚠ $\mathcal{A}_\varepsilon - \mathrm{Id}$ is 1-homogeneous: $(\mathcal{A}_\varepsilon - \mathrm{Id})[\lambda u] = \lambda(\mathcal{A}_\varepsilon - \mathrm{Id})[u]$

- We will look for an approximation of

$$\left(Lu(x)\right)^{\frac{1}{2}}$$

Consider the problem: $\boxed{|\nabla u|\,\Delta u = f}$ with $f > 0$

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- Then $|\nabla u| > 0$ and $\Delta u > 0$. Hence:

$$\boxed{|\nabla u|^{\frac{1}{2}} (\Delta u)^{\frac{1}{2}} = f^{\frac{1}{2}}}$$

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- Let us recall the identity

$$a^{\frac{1}{2}} b^{\frac{1}{2}} = \inf_{c>0} \left\{ \frac{c}{2} a + \frac{1}{2c} b \right\} \qquad \forall a, b \geq 0$$

$$\boxed{\inf_{c>0} \left\{ \frac{c}{2} |\nabla u| + \frac{1}{2c} \Delta u \right\} = f^{\frac{1}{2}}}$$

$$\boxed{|\nabla u(x)|\,\Delta u(x) = f(x)} \iff \boxed{\inf_{c>0}\left\{\frac{c}{2}|\nabla u(x)| + \frac{1}{2c}\Delta u(x)\right\} = f(x)^{\frac{1}{2}}}$$

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- $|\nabla u(x)| = \dfrac{1}{\rho}\left(\sup_{B_\rho(x)} u - u(x)\right) + O(\rho).$

- $\Delta u(x) = \dfrac{1}{\rho^2}\left(\int_{B_\rho(x)} u(y)\,dy - u(x)\right) + O(\rho^2)$

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- $$|\nabla u(x)| = \frac{1}{\eta}\Big(\sup_{B_\eta(x)} u - u(x)\Big) + O(\eta).$$
- $$\Delta u(x) = \frac{1}{\rho^2}\left(\fint_{B_\rho(x)} u(y)\,dy - u(x)\right) + O(\rho^2)$$
- Choosing $\eta = c\varepsilon^2$ and $\rho = c^{-\frac{1}{2}}\varepsilon$

$$\boxed{\underbrace{\inf_{c>0}\left\{\frac{1}{2}\sup_{B_{c\varepsilon^2}(x)} u + \frac{1}{2}\fint_{B_{c^{-\frac{1}{2}}\varepsilon}(x)} u(y)\,dy\right\}}_{\mathcal{T}_\varepsilon[u](x)} - u(x) = f(x)^{\frac{1}{2}}\,\varepsilon^2}$$

In other words:

Lemma del Teso - Rossi (2025)

Let $\phi \in C^2(B_R(x))$ be such that $Lu(x) = |\nabla u(x)|\,\Delta u(x) > 0$

$$\frac{1}{\varepsilon^2}\left(\inf_{c\in[m(\varepsilon),M(\varepsilon)]}\left\{\frac{1}{2}\sup_{B_{c\varepsilon^2}(x)} u + \frac{1}{2}\fint_{B_{c^{-\frac{1}{2}}\varepsilon}(x)} u(y)\,dy\right\} - u(x)\right) = \left(Lu(x)\right)^{\frac{1}{2}} + o_\varepsilon(1)$$

In other words:

Lemma del Toso - Rossi (2025)

Let $\phi \in C^2(B_R(x))$ be such that $Lu(x) = |\nabla u(x)| \Delta u(x) > 0$

$$\frac{1}{\varepsilon^2}\left(\inf_{c\in[m(\varepsilon),M(\varepsilon)]} \left\{ \frac{1}{2} \sup_{B_{c\varepsilon^2}(x)} u + \frac{1}{2} \fint_{B_{c^{-\frac{1}{2}}\varepsilon}(x)} u(y)\,dy \right\} - u(x) \right) = \left(Lu(x) \right)^{\frac{1}{2}} + o_\varepsilon(1)$$

Theorem del Toso - Rossi (2025)

The following are equivalent in the viscosity sense.

$$|\nabla u(x)| \Delta u(x) = f(x)$$

$$\Updownarrow$$

$$\inf_{c\in[m(\varepsilon),M(\varepsilon)]} \left\{ \frac{1}{2} \sup_{B_{c\varepsilon^2}(x)} u + \frac{1}{2} \fint_{B_{c^{-\frac{1}{2}}\varepsilon}} u(y)\,dy \right\} = u(x) + f(x)^{\frac{1}{2}} \varepsilon^2 + o(\varepsilon^2)$$

DPP FOR P-LAPLACE PROBLEMS

Theorem del Teso - Rossi (2025)

$$\begin{cases} u_\varepsilon(x) = \inf_{c\in[m,M]} \left\{ \frac{1}{2} \sup_{B_{c\varepsilon^2}(x)} u_\varepsilon + \frac{1}{2} \fint_{B_{c^{-\frac{1}{2}}\varepsilon}(x)} u_\varepsilon(y)\,dy \right\} - \varepsilon^2 f(x)^{\frac{1}{2}} & x\in\Omega \\ u_\varepsilon(x) = g(x) & x\in\mathbb{R}^d\setminus\Omega \end{cases}$$

Then

$$u_\varepsilon \longrightarrow u \quad \text{unif in } \overline{\Omega}$$

where u is the unique viscosity solution of

$$\begin{cases} |\nabla u(x)|\,\Delta u(x) = f(x) & x\in\Omega \\ u(x) = g(x) & x\in\partial\Omega \end{cases}$$

THE GAME

$$u_\varepsilon(x) = \inf_{c\in[m,M]} \left\{ \frac{1}{2} \sup_{B_{c\varepsilon^2}(x)} u + \frac{1}{2} \int_{B_{c^{-\frac{1}{2}}\varepsilon}(x)} u(y)\,dy \right\} - \varepsilon^2 f(x)^{\frac{1}{2}} \qquad x\in\Omega$$

Two players: P1 - Wants to maximize the final payoff
P2 - Wants to minimize the final payoff

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(ii)(c) If tails, the next position of the game is chosen randomly in $B_{c^{-\frac{1}{2}}\varepsilon}$.

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(ii)(c) If tails, the next position of the game is chosen randomly in $B_{c^{-\frac{1}{2}}\varepsilon}$.

(iii) Repeat the process until reaching $x_J\in\mathbb{R}^d\setminus\Omega$. Then P2 pays P1 the amount $-\varepsilon^2\sum_{j=0}^{J-1} f(x_j) + g(x_J)$.

THANK

YOU